Reading course
Western University
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No supervising professor

Fulton's *Algebraic Curves*

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Abstract

This document contains solutions to exercises from Fulton's *Algebraic Curves*. These solutions were produced while I was engaged in a reading course during the first semester of my MSc at Western University. David K. Johnson, Noah Vale, and Jaden Visscher read with me, and I owe many thanks to them for their help.

§ 1 Affine Algebraic Sets

1.1 Algebraic Preliminaries

1.4: For k infinite, the only polynomial that is zero everywhere is the zero polynomial.

Proof. Let $F \in k[X_1, \dots, X_n]$ be a polynomial such that $F(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. We proceed by induction on the number of variables n. For the case $n = 0$ the statement holds trivially. Now suppose the statement holds for $n - 1$. Then we may write $F = \sum F_i X_n^i$ for $F_i \in k[X_1, \dots, X_{n-1}]$. Now suppose that one of the F_i is not equal to zero. Then there are only finitely many roots for the polynomial $F(a_1, \dots, a_{n-1}, X_n)$, as it is univariate. However the field k is infinite so this polynomial cannot possibly vanish everywhere on k. Therefore $F = 0$. △

1.2 Affine Space and Algebraic Sets

1.3 The Ideal of a Set of Points

1.21: An ideal generated by linear polynomials is maximal and the quotient is isomorphic to the field of coefficients

Proof. Suppose $I = (X_1 - a_1, \dots, X_n - a_n)$, and let $J \supseteq I$. Then I is the ideal of the point $\mathbf{a} = (a_1, \dots, a_n)$ and $V(J) \subseteq V(I)$. However $V(I) = \{\mathbf{a}\})$, so $J = I$. That is, I is maximal. Now take the quotient $k[X_1, \dots, X_n]/I$. Now the only ideals in the quotient are the whole ring and the zero ideal, which implies that the ring is a field. The only field contained in $k[X_1, \dots, X_n]$ is k, so the quotient must be isomorphic to this field. △

Proposition 1: An algebraic set is irreducible if and only if its ideal is prime.

Proof. First the backward direction. Suppose $I(V)$ not prime. Then there is $fg \in I$ such that $f \notin I$ and $g \notin I$. Then $V(f) \cap V \cup V(g) \cap V = V$, and so V is reducible.

Now the forward direction. Suppose $V = U \cup W$ is reducible. Then for $f \in I(U) \setminus I$ and $g \in I(W) \setminus I$ we have that $fg \in I$, so I is not prime. △

1.4 The Hilbert Basis Theorem

1.22: Bijections between ideals of R/I and ideals containing I.

Proof. Part (a) asks us to prove the above statement for any ideals. First the forward direction. If J' is an ideal of R/I then $\pi^{-1}(J')$ is in particular closed under multiplication and addition of generators by the fact that π is a homomorphism. That is, it is an ideal. The fact that it contains I follows by the fact that $0 \in J'$ and $\pi^{-1}(0) = I$.

Part (b) asks us to prove that radicality, primality, and maximality are preserved. First radicality. Supposing J' is radical, we have $f^m \in J'$ implies that $f \in J'$. Then taking preimages, $\pi^{-1}(f^m) = f^m + I = (f+I)^m = \pi^{-1}(f)$, so $\pi^{-1}(J')$ is radical. Likewise, if J is radical then $\pi(J)$ is obviously radical because $\pi(f^m) = \pi(f)^m$ and $\pi(f) \in \pi(J)$. Now primality. If J' is prime then $\pi^{-1}(J') = J$ is because $\pi^{-1}(fg) = fg + I = (f+I)(g+I)$ and one of $f+I$ or $g+I$ is in the preimage. Conversely, if J is prime then multiplicativity of the homomorphism shows that J' is as well. For maximality, note that projection preserves containment.

For part (c), note that if J is finitely generated then we may write any element of J' as

$$\pi(x) = \pi\left(\sum c_\alpha y_i\right) = \sum \pi(c_\alpha)\pi(y_i)^\alpha$$

for y_i generators of J. △

1.5 Irreducible Components of an Algebraic Set

1.6 Algebraic Subsets of the Plane

1.7 Hilbert's Nullstellensatz

1.38: Bijections between algebraic subsets of an algebraic set $V = V(I)$ and radical ideals of $k[X_1, \ldots, X_n]/I$.

Proof. The Nullstellensatz demonstrates bijections between algebraic sets and radical ideals, and 1.22(a-b) (above) demonstrates the necessary bijections once one notes the inclusion-reversing properties of $I(\cdot)$ and $V(\cdot)$ as functors (J contains I exactly when $V(J)$ is a subset of $V(I)$) and recalls that irreducible varieties have prime ideals, while points has maximal ideals. △

1.8 Modules: Finiteness Conditions

1.9 Integral Elements

1.10 Field Extensions

§ 2 Affine Varieties

2.1 Coordinate Rings

2.1: Show that the map $k[X_1, \ldots, X_n]$ associating to each F a polynomial function in $\mathscr{F}(V, k)$ is a ring homomorphism with kernel $I(V)$.

Proof. Let $\varphi : k[X_1, \ldots, X_n] \to \mathcal{F}(V, k)$ assign each polynomial a polynomial function. This is obviously a ring homomorphism, and the kernel is shown to be $I(V)$ by noting that $f \to 0 \iff f(v) = 0$ for all $v \in V \iff f \in I(V)$. △

2.2: Show that there is a natural one-to-one correspondence between algebraic susbets (resp. subvarieties, resp. points) and radical ideals (resp. prime ideals, resp. maximal ideals) of $\Gamma(V)$.

Proof. This follow by recalling that $\Gamma(V) = k[X_1, \ldots, X_n]/I(V)$ and applying 1.22 and 1.38 (above). △

2.3: Show that every polynomial function on V restricts to one on $W \subseteq V$.

Proof. For part (a), note that $I_V(W)$ is the image of $I(W)$ under the quotient $k[X_1, \ldots, X_n]/I(V)$. Then obviously any polynomial on V restricts to one on W by the fact that polynomials are still polynomials under the quotient map. This map is surjective precisely because it is a quotient map $k[X_1, \ldots, X_n]/I(V) \to k[X_1, \ldots, X_n]/I(W)$ and $I(W) \supseteq I(V)$. The isomorphism follows because $I_V(W)$ is approximately $I(W)/I(V)$. △

2.4: Show that the following are equivalent: V is a point, $\Gamma(V) = k$, and $\dim_k \Gamma(V) < \infty$.

Proof. The equivalence of (ii) and (iii) are the content of 1.21 (above). Corollary 4 of the Nullstellensatz tells us that V is finite, and because it is a variety it is irreducible and hence a point. (ii) implies (iii) straightforwardly as k is a k-vector space of dimension one. △

2.2 Polynomial Maps

2.6: Show that the composition of pullbacks is pullback of the (reversed) composition, and that the composition of polynomial maps is a polynomial map.

Proof. Let $\varphi : V \to W$ and $\psi : W \to Z$. Now $\widetilde{\psi \circ \varphi}(F) = F \circ \psi \circ \varphi = \widetilde{\varphi}(F \circ \psi) = (\widetilde{\varphi} \circ \widetilde{\psi})(F)$. If φ and ψ are polynomial maps, then $\varphi = (T_1, \dots, T_n)$ and $\psi = (S_1, \dots, S_m)$, and from this it follows that $\psi \circ \varphi = (S_1(T_1, \dots, T_n), \dots, S_m(T_1, \dots, T_n))$. Now one can see that $\psi \circ \varphi$ is obviously polynomial by expanding. △

2.7: Show that the preimage of an algebraic subset is algebraic, and that if the preimage of an algebraic set is irreducible, then so long as the set lives in the image it is also irreducible.

Proof. Let $\varphi : V \to W$ be a polynomial map and X an algebraic subset of W. Let $I_W(X)$ be the ideal of X in $\Gamma(W)$ and take $Y = V(F \in \Gamma(V) \mid F \circ \varphi \in I_W(X))$. Then $\varphi(Y) = X$. The claim on irreducibility follows by the fact that if Y is reducible then there will be some $F, G \in \Gamma(V)$ such that $FG \circ \varphi \in I_W(X)$ but neither will have pullback lying in the ideal, implying that it is not prime and so X is reducible. △

2.9: Show that for a polynomial map of varieties $\varphi : V \to W$ and subvarieties V' and W' we have $\varphi(V') \subseteq W'$ implies that $\widetilde{\varphi}(I_W(W')) \subseteq I_V(V')$, and that the restriction gives a polynomial map $V' \to W'$.

Proof. Let $\varphi : V \to W$ be a polynomial map and $V' \subseteq V$, $W' \subseteq W$ subvarieties and $\varphi(V') \subseteq W'$. For part (a), we have that because $\varphi(V') \subseteq W'$, $V' \subseteq \varphi^{-1}(W')$. Then $\varphi^{-1}(W')$ is a variety by 2.7, and so we conclude that $I_V(V') \supseteq I_V(\varphi^{-1}(W'))$. Now note that any polynomial F vanishing on $\varphi^{-1}(W')$ satisfies $\widetilde{\varphi}(F) \in I_W(W')$. Therefore $I_W(W') \subseteq I_V(\varphi^{-1}(W'))$ and the result holds.

Part (b) seems more or less obvious: $\varphi(V') \subseteq W'$ means that the map is defined on all of V', and it being polynomial at every point in V means that it is, in particular, polynomial at $V' \subseteq V$. Perhaps one must use Part (a) to check that nothing is vanishing when it shouldn't. △

2.10: Show that the projection map $\mathbb{A}^n \to \mathbb{A}^r$ is polynomial.

Proof. This is the map $\varphi = (x_1, \dots, x_r)$. △

2.3 Coordinate Changes

2.14: Show that affine changes of coordinates preserve linear subvarieties of $\mathbb{A}^n$ and that any nonempty linear subvariety may be expressed as the vanishing of some $X_{m+1}, \dots, X_n$. Show also that the m before is invariant of the choice of coordinate change.

Proof. Let $V \subseteq \mathbb{A}^n(k)$ be a linear subvariety (that is, $V = V(F_1, \dots, F_r)$ for F_i linear) and let $I = I(V)$. First we show part (a). Let T be an affine change of coordinates on $\mathbb{A}^n$. Now $V^T = V(F_1^T, \dots F_r^T) = V(F_i(T_1, \dots, T_n) \mid 1 \le i \le r)$. Writing $F = \sum a_i X_i$ by linearity, we now have $F^T = \sum a_i T_i = \sum a_i \sum b_j X_j$ which is obviously a linear polynomial. Implicity we are using the fact that I^T is generated by F_i^T, which is clear. Therefore V^T is linear.

For part (b) we proceed by induction on r. For $r = 1$ and $V = V(F)$, write $F = \sum a_i X_i$. Let j be the lowest index for which $a_j \neq 0$ so we may write $T_j = (X_j - \sum_{i \neq j} a_i X_i)/a_j$ and $T_i = X_i$ for $i \neq j$. Then $F^T = F(T_1, \dots, T_n) = \sum_{i \neq j} a_i X_i + a_j(X_j - \sum_{i \neq j} a_i X_i)/a_j = X_j$. Now consider S for which $S_j = X_n$ and $S_i = X_i$ for all other $i \neq j$. Now observe that $F^{T \circ S} = F(T_1(S_1, \dots, S_n), \dots, T_n(S_1, \dots, S_n)) = (F^T)(S_1, \dots, S_n) = (F^T)^S = X_j^S = X_n$. Then the base case of induction holds. Now suppose that for $r - 1$ the statement holds and consider $V = V(F_1, \dots, F_r)$. Now $V(F_1, \dots, F_{r-1})^T = V(X_{m+1}, \dots, X_n)$ for some m and some affine coordinate change T. Then F_r^T is a polynomial in $X_{m+1}, \dots, X_n$ and we may apply the above construction to produce a map S that sends F^T to X_m and fixes all other variables. Postcomposition of T with this map then yields the desired map.

For part (c), we suppose for the sake of contradiction that $V(X_{m+1}, \dots, X_n)^T = V(X_{s+1}, \dots, X_n)$ for $m < s$. We may write $X_j = T(X_i) = T_i$ for some $s + 1 \le j \le n$ and $m + 1 \le i \le n$. However this implies that there are two X_j given by the same T_i, so in particular $T_{m+1}, \dots, T_n$ are dependent and therefore the map is not a change of coordinates. △

2.15: Let $P = (a_1, \dots, a_n)$ and $Q = (b_1, \dots, b_n)$ be distinct points in $\mathbb{A}^n$. The line through P and Q is defined to be $\{a_1 + t(b_1 - a_1), \dots, a_n + t(b_n - a_n) \mid t \in k\}$.

(a) Show that if L is the line through P and Q and T is an affine change of coordinates then $T(L)$ is the line through $T(P)$ and $T(Q)$.

(b) Show that a line is a linear subvariety of dimension one, and that a linear subvariety of dimension one is a line through any two of its points.

(c) Show that in $\mathbb{A}^2$ a line is the same thing as a hyperplane.

(d) Let $P, P' \in \mathbb{A}^2$, L_1, L_2 two distinct lines through P, L_1', L_2' two distinct lines through P'. Show there is an affine change of coordinates taking P to P' and L_i to L_i'.

Proof. Part (a): The points on $T(L)$ are of the form $T(P + t(Q - P))$. The fact that T is a composition of a linear map and a translation in each coordinate means that $T(P + t(Q - P)) = T(P) + T(t(Q - P)) = T(P) + t(T(Q) - T(P))$.

Part (b): A linear subvariety of dimension one is a linear subvariety V for which there exists a change of coordinates T such that $V^T = V(X_2, \dots, X_n)$. Then the points of the subvariety under the transformation T are given by all points of the form $(a_1, 0, \dots, 0)$. This is a line through the points $(0, \dots, 0)$ and $(1, 0, \dots, 0)$, and the fact that lines are preserved under affine coordinate changes implies that V is a line. The converse clearly holds by considering the preimage of any two points in V^T under T.

Part (c): A line is a linear subvariety of dimension one and so is defined by the vanishing of a single linear polynomial by 2.14. Therefore it is a hyperplane. Conversely, a hyperplane is defined by the vanishing of a single linear form and so by 2.14 we have that it is isomorphic to $V(X_2)$. This implies that it is of dimension one, however, and so a line.

Part (d): Consider $V(F_1, F_2)$ the intersection of L_1 and L_2. By 2.14 there is a coordinate change taking L_1 to X_1 and L_2 to X_2, as they are linearly independent forms. Similarly, there is a coordinate change taking L_1' and L_2' to X_1 and X_2, respectively. Composing the former map with the inverse of the latter map yields a coordinate change taking L_1 to L_1', L_2 to L_2', and $P = V(L_1, L_2)$ to $P' = V(L_1', L_2')$. It is worth noting that because L_1 and L_2 are distinct they are linearly independent so our maps cannot send them to the same X_i. △

2.4 Rational Functions and Local Rings

2.18: Show that there is a natural one-to-one correspondence between prime ideals of $\mathscr{O}_P(V)$ and subvarieties of V passing through P.

Proof. Suppose I is prime in $\mathscr{O}_P(V)$ so that $I \cap \Gamma(V)$ is also prime ($fg \in I \cap \Gamma(V)$ implies that wlog $f \in I$ but $fg \in I \cap \Gamma(V)$ implies wlog f is polynomial and clears g's denominator). By the proof of Proposition 3 we have that $I \cap \Gamma(V)$ generates I. Now apply the bijection between prime ideals in $\Gamma(V)$ and subvarieties so that $I \cap \Gamma(V)$ gives a subvariety, and note that $P \in V(I \cap \Gamma(V))$ by the fact that every $h \in I$ is defined at P. △

2.21: Show that a polynomial map of affine varieties induces a unique ring homomorphism $\mathscr{O}_Q(W) \to \mathscr{O}_P(V)$. Show that the maximal ideal of the codomain contains the image of the domain's maximal ideal.

Proof. Suppose $\varphi : V \to W$ is a polynomial map such that $P \in V$ has $\varphi(P) = Q$. Now consider the map on coordinate rings $\widetilde{\varphi} : \Gamma(W) \to \Gamma(V)$. We wish to show that $\widetilde{\varphi}$ extends uniquely to a homomorphism $\mathscr{O}_Q(W) \to \mathscr{O}_P(V)$. We define such a homomorphism by $\widetilde{\varphi}(f/g) = \widetilde{\varphi}(f)/\widetilde{\varphi}(g)$. This map respects the equivalence relations as if $f/g \sim p/q$ then $fq - pg = 0$ and so $\widetilde{\varphi}(fg - pq) = \widetilde{\varphi}(f)\widetilde{\varphi}(q) - \widetilde{\varphi}(g)\widetilde{\varphi}(p) = 0$ and obviously restricts to the map on coordinate rings. The fact that it is a homomorphism follows by the fact that it is a homomorphism on coordinate rings and the definition of the field of fractions. Uniqueness is clear by construction. In particular, $\widetilde{\varphi}(\mathfrak{m}_Q(W)) \subseteq \mathfrak{m}_P(V)$ as if $f/g \in \mathfrak{m}_Q(W)$ vanishes at Q then in particular $f(Q) = 0$ implying that $f(\varphi(P)) = 0$ and so $\widetilde{\varphi}(f)(P) = 0$. △

2.22: Show that for an affine change of coordinates on $\mathbb{A}^n$, the map described above is an isomorphism and induces an isomorphism on varieties before and after the change of coordinates.

Proof. Suppose $T : \mathbb{A}^n \to \mathbb{A}^n$ is an affine change of coordinate with $T(P) = Q$. Notably the coordinate rings are isomorphic and so by construction of $\widetilde{T}$ the local rings are isomorphic as well. If V is a subvariety of $\mathbb{A}^n$ such that $P \in V^T$, then the restriction to

$\mathscr{O}_Q(V)$ has image given by F^T/G^T for all $F/G \in \mathscr{O}_Q(V)$. However this is precisely the field of fractions of V^T, so long as $P \in V^T$ to ensure that we restrict to where F^T is defined. $\triangle$

2.5 Discrete Valuation Rings

2.23: Show that the order function on K is independent the choice of a uniformizing parameter.

Proof. If t is a uniformizing parameter for R then any other uniformizing parameter is of the form $s = ut$ for $u \in R^\times$ a unit. Then if the order of $z \in K = \mathrm{Frac}(R)$ is n we have $z = vt^n$ for some $v \in R^\times$. We may then write $z = vt^n = v(u^{-1}s)^n = ws^n$ for $w = vu^{-n}$ a unit. This expression in terms of the uniformizing parameter is unique, and so the order is fixed. $\triangle$

2.24: Show that the affine line has local ring given by a DVR.

Proof. First we show part (a). Let $f/g \in \mathfrak{m}_a(V) \subset \mathscr{O}_a(\mathbb{A}^1_k)$. Then f/g vanishes at a, implying that $f = h\cdot(X-a)$ for some $h \in \Gamma(V)$, and we may write $f = h\cdot(X-a)^n$ with h not divisible by $(X-a)$. In this case f/g permits an expression of the form $f/g = h/g\cdot(X-a)^n$, and h/g is a unit because neither is divisible by $(X-a)$. Importantly this unit is unique by construction, and so $\mathscr{O}_a(V)$ is a DVR.

Now part (b). We let $\mathscr{O}_\infty(V) = \{f/g \mid \deg(g) \geq \deg(f)\}$. Let $f/g \in \mathscr{O}_\infty(V)$ and suppose that we take a representative with such that X is not a factor of both f and g. Note that f/g is a unit if and only if $\deg f = \deg g$, as then $g/f \in \mathscr{O}_\infty(V)$. Now for any f/g we may take $X^n f/g$ to be a unit for n sufficiently large, implying that $f/g = X^n f/g\cdot(1/X)^n$. This expression is unique by construction, and so $\mathscr{O}_\infty(V)$ is a DVR. $\triangle$

2.26: Show that the maximal ideal of a DVR contains any element of R or its inverse. Show that if S is a DVR containing R, contained in the quotient field of R, and whose maximal ideal contains that of R, then $S = R$.

Proof. Part (a): Let $z \in K$ such that $z \notin R$. Then the order of z is negative, implying that $z = ut^n$ for $n < 0$. Now take an inverse to see that $z^{-1} = u^{-1}t^{-n}$, so in particular z has positive order implying that $z \in \mathfrak{m}$.

Part (b): Suppose $\mathfrak{m}_R \subseteq \mathfrak{m}_S$. Suppose $z \in S - R$. Then $z^{-1} \in \mathfrak{m}_R$ by (a), so $z^{-1} \in \mathfrak{m}_S$. However $z \in S$ so $zz^{-1} \in \mathfrak{m}_S$ which is a contradiction. Therefore $S = R$. $\triangle$

2.28: Show that a discrete valuation on a field gives a DVR by the elements of K with nonnegative valuation.

Proof. Let φ be a valuation on K. It is clear that R is a ring and $\mathfrak{m}$ is an ideal. Observe that $\mathfrak{m}$ is the set of non-units in R, and so R has a unique maximal ideal containing every proper ideal of R by the Lemma in 2.4, which is equal to $\mathfrak{m}$. That is, R is local. Suppose $x \in R$. We wish to show that $x = ut^n$ for some irreducible $t \in R$, some unit $u \in R^\times$, and some $n \geq 0$. Fix some irreducible $t \in \mathfrak{m}$ of minimal degree. Then any nonzero $y \in R$ has $0 \leq \varphi(yt^{-m}) < \varphi(t)$ for some m. This implies that yt^{-m} is a unit, as $\varphi(t)$ is minimal. Equivalently, $y = ut^m$ and this is a unique expression by construction.

The converse is obvious. $\triangle$

2.29: Show that if the order of two elements is different, their sum is equal to the minimum of the two orders. Show that if a collection of elements of K has an element of strictly minimal order then their sum is nonzero.

Proof. Part (a): If $\varphi(a) < \varphi(b)$ then $\varphi(a+b) \geq \min(\varphi(a), \varphi(b)) = \varphi(a)$.

Part (b): If there exists i such that $\varphi(a_i) < \varphi(a_j)$ for all $j \neq i$ then $\varphi(a_1 + \cdots + a_n) = \varphi(a_i) \neq \infty$, so $a_1 + \cdots + a_n \neq 0$. $\triangle$

2.30: Let R be a DVR with quotient field K, $k \subseteq R$ a field, and an isomorphism $k \to K \to R/\mathfrak{m}$. Show that for any $z \in R$ there is unique $\lambda \in k$ such that $z - \lambda \in \mathfrak{m}$ and show that any $z \in R$ admits an expression $z = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n + z_n t^{n+1}$ for $\lambda_i \in k$ and $z_n \in R$.

Proof. Part (a): Let $z \in R$. If $z \in \mathfrak{m}$ then $\lambda = 0$. Suppose $z \notin R$ and note that the coset $z + \mathfrak{m} \in k$ has additive inverse $-\lambda$ such that $z - \lambda = 0 \in k$. However $z - \lambda = 0 \in k$ implies that the preimage lives in $\mathfrak{m}$.

Part (b): To establish existence we proceed by induction on n. For the case $n = 0$ we write $z - \lambda \in \mathfrak{m}$ which implies that $z = \lambda \in k$, which in particular implies that $z = \lambda \in R$. Now suppose the result holds for $n = k - 1$, so there are $\lambda_0, \dots, \lambda_{k-1} \in k$ and $z_{k-1} \in R$ for which $z = \lambda_0 + \cdots + \lambda_{k-1}t^{k-1} + z_{k-1}t^k$. Now we may write $z_{k-1} - \lambda_k \in \mathfrak{m}$ so $z = \lambda_k \in R$ as in the base case. Now $z = \lambda_0 + \cdots + \lambda_k t^k$ with $z_k = 0$. So the inductive step holds.

For uniqueness, consider the difference of two such expressions $0 = z - z = (\lambda_0 - \mu_0) + \cdots + (\lambda_n - \mu_n)t^n + (z_n - y_n)t^{n+1}$. Now note that the order of the term $\lambda_0 - \mu_0$ is minimal in this sum, so by 2.29 the sum cannot be zero. However this implies that $0 \neq 0$ so we have a contradiction, establishing uniqueness. △

2.6 Forms

2.35: Count the dimension of forms of degree d as a k-vector space and show that sequences of nonzero linear forms (some not scalar multiples of others) generate the space of d-forms on two variables.

Proof. Part (a): A monomial of degree d in $R[X, Y]$ is of the form X^iY^j such that $i + j = d$. There are $d + 1$ pairs satisfying this, as can be seen by noting that $i = d - j$ is a solution for all $0 \leq j \leq d$. Similarly, in $R[X, Y, Z]$ we have as many monomials as triples $i + j + k = d$ and writing $i = d - j - k$ we see that the number of such triples is given by $\sum_{j=0}^{d} \sum_{k=0}^{i} 1 = \sum_{j=0}^{d} j = (d + 1)(d + 2)/2$.

Part (b) is obvious by the definition of forms.

Part (c): Incomplete △

2.7 Direct Products of Rings

2.38: Show that the direct sum adds dimension.

Proof. The direct product is obviously generated as a ring by the generators of each R_i, and so there are $\dim R_i$ generators for each component. Components are independent, so there are $\sum \dim R_i$ generators. Equivalently, the dimension of the product is the sum of dimensions. △

2.8 Operations with Ideals

2.39: Show that ideal multiplication is distributive and associative.

Proof. Part (a): $(I_1 + I_2)J = \{a + b \mid a \in I_1, b \in I_2\} \cdot J = \{(a + b)c \mid a \in I_1, b \in I_2, c \in J\} = \{ac + bc \mid \dots\} = I_1J + I_2J$.

Part (b): $(I_1 \dots I_m)^n = (\{a_1 \dots a_m \mid a_i \in I_i\})^n = \{(a_{1,1} \dots a_{1,n}) \dots (a_{1,1} \dots a_{1,m} \mid a_{i,j} \in I_i\} = I_1^n \dots I_m^n$. △

2.40: Show that for I, J comaximal we have $I + J^2 = R$, I^m and J^n comaximal, and that products of comaximal ideals agree with their intersections, even when taking powers.

Proof. Part (a): If I, J are comaximal then $I + J = R$. Then $1 = a + b$ for $a \in I$, $b \in J$. Now $1 = 1^2 = a^2 + 2ab + b^2 = a(a + 2b) + b^2$. Now $a(a + 2b) \in I$ and $b^2 \in J^2$, so in particular $I + J^2 = R$. This naturally generalizes: $1 = 1^m = af + b^m$ for $f = \sum_{i+j=m} a^i b^j$ where the sum ranges over $0 \leq i < j \leq m - 1$. This shows that I and J^m are radical, and because our rings are commuative $I + J = J + I$ so we may perform the same process to obtain that $J^m + I^n = R$.

Part (b): The Lemma states that I, J comaximal implies that $IJ = I \cap J$, and part (a) shows that any powers of I and J are comaximal. Applying this to $J_i = \bigcap_{i \neq j} I_j$ we note that $I_1^n \cap \bigcap_{j=2}^{m} I_j^n = I_1^n \cdot \bigcap_{j=2}^{m} I_j^n = \cdots = I_1^n \dots I_m^n = (I_1 \dots I_m)^n = (I_1 \cap \cdots \cap I_m)^n$. △

2.41: Show that if I is in the radical of an ideal, then some power of I is in the ideal (ChatGPT helped solved this one).

Proof. Suppose $I \subseteq \sqrt{J}$. Then for any $a \in I$ there is some minimal $n \geq 0$ such that $a^n \in J$. For $g_1, \dots, g_m$ generators of I we write $a = \sum b_i g_i$. Then $a^n = \sum b_1^{i_1} \dots b_m^{i_m} g_1^{i_1} \dots g_m^{i_m}$ where the sum is over all $i_1, \dots, i_m$ such that sum $\sum i_j = n$. Let m_i be the minimal power of g_i such that $g_i^{m_i} \in J$. Taking $n = \sum m_i$, we have that a^n is a sum of elements of the form $g_1^{i_1} \dots g_m^{i_m}$. Now because n is sufficiently large we have that, for any such product, $\sum i_j = n$

implying that at least one $i_j \geq m_j$. This implies that $g_j^{m_j}(g_1^{i_1} \dots g_j^{i_j - m_j} \dots g_m^{i_m}) \in J$, as J is an ideal. Therefore every product of the form $g_1^{i_1} \dots g_m^{i_m}$ such that $\sum i_j = n$ is in J, implying that $I^n \subseteq J$. △

2.42: Show that there are natural homomorphisms between quotients: $R/I \to R/J$ for $I \subseteq J$ and $R/I \to S/IS$.

Proof. Part (a): If $I \subseteq J$, then R/I has a natural map onto R/J given by the identity map defined on cosets of I. For $a, b \in R$, $a \sim b$ in R/I if and only if $a - b \in I$ which implies that $a - b \in J$ so $a \sim b$ in R/J. Then the fact that we have a homomorphism follows by the fact that R/J is a ring.

Part (b): If $R \subseteq S$ is a subring then we have the natural homomorphism $R/I \to R/IS$ by the fact that $I \subseteq IS$. Now there is another natural injective homomorphism $R/IS \to S/IS$ given by the identity, and the composition of these two gives the desired natural map. △

2.43: Show that the maximal ideal of the local ring of a point $P = (0, \dots, 0)$ is $(X_1, \dots, X_n)\mathscr{O}$ and that $\mathfrak{m}^r = I^r\mathscr{O}$.

Proof. The maximal ideal of a point is given by all non-units of $\mathscr{O}_P$, i.e. those vanishing at P. A polynomial with no constant term must vanish at P, and I consists of precisely these polynomials. Therefore and $f \in I\mathscr{O}$ has numerator vanishing at P, implying that $f \in \mathfrak{m}$. In fact all these statements are "if-and-only-if," so the result holds. △

2.44: Quotients of defining ideals in the local ring of a point.

Proof. Incomplete △

2.45: Show that the intersection of two ideals is empty if and only if their defining ideals are comaximal.

Proof. Observe: $V(I) \cap V(J) = V(I + J) = \emptyset$ if and only if $I + J = R$. △

2.46: Show that $\dim_k(k[X, Y]/(X, Y)^n) = n(n + 1)/2$.

Proof. The power ideal I^n consists of all degree n polynomials in X and Y. Then $k[X, Y]/I^n$ is the set of all degree $d < n$ polynomials, which has dimension $n(n + 1)/2$. △

2.9 Ideals with a Finite Number of Zeros

No starred exercises.

2.10 Quotient Modules and Exact Sequences

2.48: Show that the image and kernel of an R-module homomorphism are R-modules and that they form an exact sequence.

Proof. The kernel and image are, in particular, subgroups for any group homomorphism, and the R-action obviously extends. To see exactness, note that the inclusion $\ker\varphi \subseteq M$ has image equal to the the kernel of φ (big surprise). Therefore the sequence has im $\subseteq = \ker\varphi$. △

2.49: Show exactness for a number of natural sequences.

Proof. Part (a): Consider the map $\overline{\varphi}$ that takes cosets $a + N \in M/N$ to elements $\varphi(a) \in M'$. This is well-defined because $\varphi(N) = 0$. Then $\overline{\varphi} \circ \pi(a) = \overline{\varphi}(a + N) = \varphi(a)$. Uniqueness follows by construction: Any other map factoring φ must have that $\overline{\varphi}(\pi(a)) = \varphi(a)$, so in particular cosets of a must go to $\varphi(a)$.

Part (b): If $P \subseteq N \subseteq M$ our natural homomorphisms are achieved by the same argument as in 2.42. Now

$$0 \to N/P \xrightarrow{\subseteq} M/P \xrightarrow{\varphi} M/N \to 0$$

has $\operatorname{im}\varphi = M/N$. Then $\ker\varphi = \{a \in M/P \mid a \in N\} = \{a \in N/P\} = \text{im } \subseteq$, so the sequence is exact.

Part (c): This is a consequence of (b) when $P \subseteq N \subseteq M$ are finite-dimensional vector spaces and an application of the rank-nullity theorem.

Part (d): Ideals are obviously modules. Then this follows from (b) taking $I = P \subseteq J = N \subseteq R = M$.

Part (e): This is the situation of (d) with $I = \mathfrak{m}^n \supseteq J = \mathfrak{m}^{n+1}$ and $R = \mathscr{O}$. △

2.50: Show that a DVR as in 2.30 has $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$, $\dim_k(R/\mathfrak{m}^n) = n$, and $\operatorname{ord} z = \dim_k(R/\mathfrak{m}^n)$.

Proof. Part (a): Recall that $\mathfrak{m} = (t)$. Then $\mathfrak{m}^n = (t^n)$. Now note that $\mathfrak{m}^n/\mathfrak{m}^{n+1} = t$. Therefore $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$.

Part (b): Writing $\mathfrak{m}^n = (t^n)$ we note that any element $z = ut^k$ of order greater than or equal to n lies in $\mathfrak{m}^n$, and so is mapped to zero by the quotient homomorphism. Therefore the only remaining terms are those of order $k < n$, yielding an n-dimensional vector space.

Part (c): If $(z) = \mathfrak{m}^n$ we use the fact that $\mathfrak{m}^n$ is generated by t^n for t a uniformizing parameter to see that $z = xt^n$ for some $x \in R$. However we also have that $zy = t^n$, so $z = xyz$ which implies that $xy = 1$ (recall that R is a domain). Now we see that z is of order n and so the result follows. △

2.52: Let N, P be submodules of M. Show that the subgroup $N + P$ is a submodule of M. Show that there is a natural R-module isomorphism of $N/N \cap P$ onto $(N + P)/P$.

Proof. We have a natural action of R on $N + P$ by $r(n + p) = rn + rp$. This gives the subgroup $N + P$ an R-module structure by the fact that each of N and P are R-modules. The natural R-module isomorphism sends a coset $n + N \cap P \mapsto n + P$. △

2.53: Incomplete

2.11 Free Modules

No starred exercises.

§ 3 Local Properties of Plane Curves

3.1 Multiple Points and Tangent Lines

3.1: Prove that in the above examples the only multiple point on each of C, D, E, and F is $P = (0, 0)$.

Proof. Curve C: The lowest degree form in C is $Y^2 = Y \cdot Y$, which is a double point. For any point (a, b) we consider $T = (X + a, Y + b)$ so that

$$C^T = (Y + b)^2 - (X + a)^3 = Y^2 + 2Yb - X^3 - 3aX^2 - 3a^2X$$

has (a, b) at the origin. The leading form is of degree one, so the point (a, b) is simple. A similar argument holds for the remaining curves. △

3.5: Supposing $\operatorname{char} k = 0$, show that $m = m_P(F)$ is the smallest integer such that, for some $i + j = m$, $\frac{\partial^m F}{\partial x^i \partial y^j}(P) \neq 0$. Find an explicit description for the leading form for F at P in terms of these derivatives.

Proof. It suffices to show the statement for $P = (0, 0)$. We observe that this partial derivative $\frac{\partial^m F}{\partial x^i \partial y^j}$ will, for choices of i, j given by the powers of X and Y in the monomials of the leading form of F, give a polynomial with a nonzero constant term (this is by the characteristic requirement). If there were some smaller integer that satisfied this criteria we would have a smaller degree leading form, a contradiction. The leading form, then, has the coefficient of X^iY^j given by $\frac{\partial^m F}{\partial x^i \partial y^j}(P)$. △

§ 4 Projective Varieties

4.1 Projective space

4.2: Let $F \in k[X_1, \ldots, X_{n+1}]$ for k infinite. Writing $F = \sum F_i$ for F_i a degree i form, suppose $F(P) = 0$ for all representatives of $P \in \mathbb{P}^n(k)$. Show that each $F_i(P) = 0$.

Proof. Fix some point P. $F(\lambda P) = F(P) = 0$ for all λ. That is, $F(\lambda P)$ is a polynomial in λ that is zero for all choices of λ, which implies it is the zero polynomial because we are in an infinite field (Exercise 1.4). Therefore the polynomial zero coefficients, so $F_i(P) = 0$ for all i. △

4.2 Projective Algebraic Sets

4.4: Let I be a homogeneous ideal in $k[X_1, \dots, X_{n+1}]$. Show that I is prime if and only if for any forms $F, G \in k[X_1, \dots, X_{n+1}]$, $FG \in I$ implies that $F \in I$ or $G \in I$.

Proof. ($\Longleftarrow$) We prove the contrapositive. Suppose that for some forms F, G we have that $FG \in I$ but $F \notin I$ and $G \notin I$. These forms are then a counterexample to primality for I, so I is not prime.

($\Longrightarrow$) Suppose I is prime. Then for any $FG \in I$ we have $F \in I$ or $G \in I$. We know that I contains all of its forms, so as a special case we have the previous statement for forms. $\triangle$

4.9: Let I be a homogeneous ideal in $k[X_1, \dots, X_{n+1}]$ and

$$\Gamma = k[X_1, \dots, X_{n+1}]/I.$$

Show that the forms of degree d in Γ form a finite-dimensional vector space over k.

Proof. The space of degree d forms in $k[X_1, \dots, X_{n+1}]$ is finite-dimensional by considering all degree d monomials. Passing to a quotient of this space does not increase dimension, so the claim holds. $\triangle$

4.11: Suppose $V = V(H_1, \dots, H_r)$ is a linear subvariety of $\mathbb{P}^n$.

(a) Show that if T is a projective change of coordinates then V^T is a linear subvariety.

(b) Show that there is a projective change of coordinates taking V to $V^T = V(X_{m+2}, \dots, X_{n+1})$.

(c) Show that the m appearing in (b) is independent of the choice of T.

Proof. Repeat the proof of 2.14 with the knowledge that, because we are dealing here with linear *forms* as opposed to linear polynomials, the arguments follow in $\mathbb{P}^n$. $\triangle$

4.12: Let $H_1, \dots, H_m$ be hyperplanes in $\mathbb{P}^n$, $m \leq n$. Show that $H_1 \cap \cdots \cap H_m \neq \emptyset$.

Proof. Let α_i be the linear form defining H_i. Then the intersection $H_1 \cap \cdots \cap H_m = V(\alpha_1, \dots, \alpha_m) = V$. Now consider the cone over V, $C(V)$. This is an intersection of n hyperplanes through the origin in $\mathbb{A}^{n+1}$, so in particular their intersection is not the origin, it is a line. This yields that there is projective point in V. $\triangle$

4.13: Define the line through two distinct projective points $P = [a_1 : \cdots : a_{n+1}]$ and $Q = [b_1 : \cdots : b_{n+1}]$ to be the set

$$L = \{[\lambda a_1 + \mu b_1 : \cdots : \lambda a_{n+1} + \mu b_{n+1}] \mid \lambda, \mu \in k, \lambda \neq 0 \text{ or } \mu \neq 0\}.$$

(a) Show that if L is the line through P and Q and T is a projective change of coordinates then $T(L)$ is the line through $T(P)$ and $T(Q)$.

(b) Show that a line is a linear subvariety of dimension one, and that a linear subvariety of dimension one is a line through any two of its points.

(c) Show that in $\mathbb{P}^2$ a line is the same thing as a hyperplane.

(d) Let $P, P' \in \mathbb{P}^2$, L_1, L_2 two distinct lines through P, L_1', L_2' two distinct lines through P'. Show there is an affine change of coordinates taking P to P' and L_i to L_i'.

Proof. Part (a): Suppose T is a projective change of coordinates, so it is given by linear forms $T = (T_1, \dots, T_{n+1})$. Then $T_i(\lambda a_i + \mu b_i) = \lambda T_i(a_i) + \mu T_i(b_i)$. This holds for each coordinate so we find that $T(L)$ is a line through $T(P)$ and $T(Q)$ (this is well-defined wrt to the equivalence relation on $\mathbb{P}^n$ by linearity).

Part (b): For a linear subvariety of dimension one we have that there is a projective change of coordinates taking V to $V(X_3, \dots, X_{n+1})$, which is a projective line through the points $[1 : 0 : \cdots : 0]$ and $[0 : 1 : 0 : \cdots : 0]$ by definition. Then the inverse of this linear subvariety is a line through the inverses of each points, so the linear subvariety itself is a line. Conversely, we may choose a map taking P to the former projective point and Q to the latter which gives the claim.

Part (c): A hyperplane is the vanishing of a single linear form, so by 4.11 we have that there is a coordinate change taking it to $V(X_3)$. However this makes it a linear subvariety of dimension one, so it is a line. The converse follows similarly.

Part (d): Two distinct lines each have coordinate changes taking each to $V(X_3)$, and we may take another coordinate change to send $V(X_3)$ to $V(X_2)$. In fact, because these lines are distinct there does not exist a coordinate change taking both to $V(X_3)$, so instead we choose a coordinate change taking L_1 to $V(X_3)$ and L_2 to $V(X_2)$. Then there is a coordinate change (the inverse of the one promised by 4.11) that takes L_1' to $V(X_3)$ and likewise for L_2' and $V(X_2)$. The composition of these coordinate changes gives the required maps, and the fact that P lives in the interesction of these lines gives that P maps to P'. △

4.14: Let P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) be three points in $\mathbb{P}^2$ not lying on a line. Show that there is a change of coordinates such that $T(P_i) = Q_i$. Extend this to $n+1$ points in $\mathbb{P}^n$ not lying on a hyperplane. (Solution due to David)

Proof. Consider the matrix $A = \begin{pmatrix} P_1 & P_2 & P_3 \end{pmatrix}$ with the P_i taken as column vectors. This matrix is invertible because the three points do not lie on a line, and so taking $B = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix}$ we have that BA^{-1} is the required linear transformation. The same construction works for points on a hyperplane. △

4.15: Show that two distinct lines in $\mathbb{P}^2$ intersect in a point.

Proof. By 4.13 we have that lines in $\mathbb{P}^2$ are hyperplanes, so problem 4.12 tells us that their intersection is nonempty, so it contains at least one point. If it contained two points (or more) then we may form the line through those two points to find that the intersection of the two distinct lines is a line, implying that the two lines are the same. △

4.16: Let L_1, L_2, L_3 (resp. M_1, M_2, M_3) be lines in $\mathbb{P}^2(k)$ that do not all pass through a point. Show there is a projective change of coordinates taking L_i to M_i.

Proof. The pairwise intersection of any two lines $L_i \cap L_j$ (resp. $M_i \cap M_j$) yields a point on both by 4.15. Then we have three points given by the three pairwise intersections, so we apply 4.14 to map these to the points given by the $M_i \cap M_j$. However this maps two points on each L_i to two points on each M_i, and because a line is determined by two points we see that these lines are mapped to one another. △

4.17: Let z be a rational function on a projective variety V. Show that the pole set of z is an algebraic subset of V.

Proof. Recall that a rational function on V is an element $f/g \in \operatorname{Frac}\Gamma(V)$ such that $\deg f = \deg g$. The pole set is all the zeros of g, which is really a coset $g + I(V)$ and so the ideal generated by g contains I. Therefore it corresponds to an algebraic subset. Maybe this needs more care? △

4.18: Let $H = V(\sum a_i X_i)$ be a hyperplane in $\mathbb{P}^n$. Note that $(a_1, \ldots, a_{n+1})$ is determined by H up to a constant multiple.

(a) Show that assigning $[a_1 : \cdots : a_{n+1}]$ to H gives a natural bijection between hyperplanes in $\mathbb{P}^n$ and points in $\mathbb{P}^n$.

(b) If $P \in \mathbb{P}^n$, denote by P^* the corresponding hyperplane and conversely let H^* denote the point ocrresponding to a hyperplane H. Show that $P^{**} = H$ and $H^{**} = P$. Show that $P \in H$ if and only if $H^* \in P^*$.

Proof. Part (a): For surjectivity, note that any point determines a linear form by taking its coordinates as the coefficients. To see that the map is injective, suppose that two hyperplanes determine the same point. Then their coefficients differ by a nonzero constant multiple, however this implies that their vanishing sets are the same, so they are the same hyperplane.

Part (b): First, $P^{**} = (P^*)^* = V(\sum a_i X_i)^* = [a_1 : \cdots : a_{n+1}] = P$. Likewise, $H^{**} = (H^*)^* = ([a_1 : \cdots : a_{n+1}])^* = V(\sum \lambda a_i X_i) = H$. Suppose that $P = [b_1 : \cdots : b_{n+1}] \in H = V(\sum a_i X_i)$. This occurs if and only if $\sum a_i b_i = 0$, so $P^* = V(\sum b_i X_i)$ has $H^* = [a_1 : \cdots : a_{n+1}] \in V(\sum b_i X_i) = P^*$. △

4.3 Affine and Projective Varieties

4.19: If $I = (F)$ is the ideal of an affine hypersurface, show that $I^* = (F^*)$.

Proof. I^* is the *homogenization* of I given by $I^* = \{F^* \mid F \in I\}$. If $I = (F)$ then $I^* = \{(GF)^* \mid G \in k[X_1, \ldots, X_n]\}$ and $(GF)^* = G^* F^* \in (F^*)$, so $I^* \subseteq (F^*)$. The converse is obviously true by the fact that $F \in I$. △

4.22: Suppose V is a variety in $\mathbb{P}^n$ and $V \supset H_\infty$. Show that $V = \mathbb{P}^n$ or $V = H_\infty$. If $V = \mathbb{P}^n$, $V_* = \mathbb{A}^n$, while if $V = H_\infty$, $V_* = \emptyset$.

Proof. Suppose $V \supset H_\infty$. Then any $[x_1 : \cdots : x_n, 0] \in V$ for all x_i. If V contains no other points then $V = H_\infty$. If V contains any points not of the above form, then in particular it must contain a point of the form $[x_1 \; dots \; : x_{n+1}]$ for $x_{n+1} \neq 0$. Now the irreducibility of V implies that it must contain all points with $x_{n+1} \neq 0$, or else we could write V as the union of H_∞ and some other subvariety. △

4.23: Describe all subvarieties in $\mathbb{P}^1$ and $\mathbb{P}^2$.

Proof. First we consider $\mathbb{P}^1$. Suppose $V \subseteq \mathbb{P}^n$ is a variety. If V contains H_∞ then $V = H_\infty$ or $V = \mathbb{P}^1$. If V does not contain H_∞, then $V_* \subseteq \mathbb{A}^1$ is a one-dimensional variety, so a finite set of points (or the entire affine line). So every variety in $\mathbb{P}^1$ is one of: The projective closure of a variety in $\mathbb{A}^1$, H_∞, or $\mathbb{P}^1$.

Now for $\mathbb{P}^2$. Again, if V contains H_∞ then it is either H_∞ or $\mathbb{P}^2$. By the same argument as above we see that every variety in $\mathbb{P}^2$ is the projective closure of an affine plane curve, H_∞, or $\mathbb{P}^2$. △

4.24: Let $P = [0 : 1 : 0] \in \mathbb{P}^2(k)$. Show that the lines through P consist of the following:

(a) The "vertical" lines $L_\lambda = V(X - \lambda Z) = \{[\lambda : t : 1] \mid t \in k\} \cup \{P\}$.

(b) The line at infinity $L_\infty = V(Z) = \{[x : y : 0] \mid x, y \in k\}$.

Proof. It is clear that the line at infinity and the vertical lines contain P, so we must check that the only lines containing P that are not at infinity are the vertical lines. Suppose L is a line through $P = [0 : 1 : 0]$ and let $Q = [b_1 : b_2 : b_3]$ be a point anywhere else on the line. The point Q may not lie on the line at infinity or else this line is precisely the line at infinity, so suppose $b_3 \neq 0$. The line is of the form $L = \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] \mid \lambda, \mu \in k \text{ not both equal to zero}\}$. Note that P occurs when $\mu = 0$, so we may write $L - P = \{[\mu b_1 : \lambda + \mu b_2 : \mu b_3] \mid \mu \neq 0\}$. Now the line has representatives of the form $[\mu b_1/\mu b_3 : (\lambda + \mu b_2)/\mu b_3 : 1]$, and so $L - P = L_\lambda - P$ so we are done. △

4.25: Let $P = [x : y : z] \in \mathbb{P}^2$.

(a) Show that $\{(a, b, c) \in \mathbb{A}^3 \mid ax + by + cz = 0\}$ is a hyperplane in $\mathbb{A}^3$.

(b) Show that for any finite set of points in $\mathbb{P}^2$ there is a line not passing through any of them.

Proof. Part (a): This is obvious: The set is the orthogonal complement of a dimension one object (the line $(\lambda x, \lambda y, \lambda z)$), and so has codimension one.

Part (b): Each point in $\mathbb{P}^2$ gives a hyperplane in $\mathbb{A}^3$ by the above. A projective line is of the form $[\lambda a_1 + \mu b_1 : \lambda a_2 + \mu b_2 : \lambda a_3 + \mu b_3]$ which, taken as a set in $\mathbb{A}^3$, also gives a hyperplane, as it is the span of two distinct lines. The criteria that the line given by $[a_1 : a_2 : a_3]$ and $[b_1 : b_2 : b_3]$ contains a given point $[x : y : z]$ is now equivalent to the hyperplane spanned by (a_1, a_2, a_3) and (b_1, b_2, b_3) being orthogonal to the hyperplane orthogonal to (x, y, z). Now suppose that I have a finite set of points in $\mathbb{P}^2$. Consider the hyperplanes $H_1, \ldots, H_n$ associated to this point set, and note that any vector not in a hyperplane H_i is normal to a hyperplane not orthogonal to H_i. There exists a vector not contained in any H_i by the fact that that we are taking a union of finitely many codimension one subspaces, and this vector will yield a hyperplane corresponding to a line not passing through any of the points, as it is not orthogonal to any of the associated hyperplanes and therefore does not contain any of their normals. △

4.4 Multiprojective space

4.26:

(a) Define maps $\varphi_{i,j} : \mathbb{A}^{n+m} \to U_i \times U_j \subset \mathbb{P}^n \times \mathbb{P}^m$. Using $\varphi_{n+1,m+1}$, define the "biprojective closure" of an algebraic set in $\mathbb{A}^{n+m}$. Prove an analogue of Proposition 3 of 4.3.

(b) Generalize part (a) to maps $\varphi : \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_r} \times \mathbb{A}^m \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$. Show that this yields a correspondence between $\{\text{nonempty affine varieties in } \mathbb{A}^{n_1+\cdots+m}\}$ and $\{\text{varieties in } \mathbb{P}^{n_1} \times \cdots \times \mathbb{A}^m \text{ that intersect } U_{n_1+1} \times \cdots \times \mathbb{A}^m\}$. Show that this correspondence preserves function fields and local rings.

Proof. Recall that the *projective closure* of an affine variety was given by $V^* = V(I^*) = V((F^* \mid F \in I))$, where F^* is the homogenization of F. In order to define the *biprojective closure*, we define the *bihomogenization* of a polynomial $F \in k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ as follows: Let d_1, d_2 be the degrees of F in X and Y respectively (we assume there are no monomials containing both X_is and Y_is).

$$F^* = X_{n+1}^{d_1} F(X_1/X_{n+1}, \dots, X_n/X_{n+1}, 0, \dots, 0) + Y_{n+1}^{d_2} F(0, \dots, 0, Y_1/Y_{n+1}, \dots, Y_n/Y_{n+1}).$$

Then the biprojective closure of an affine variety is given by taking the bihomogenization of its defining ideal and considering the projective variety defined by this bihomogenized ideal.

Part (b): As above, for a polynomial $F \in k[\mathbf{X}_{n_1}, \mathbf{X}_{n_2}, \dots, \mathbf{X}_m]$ we define the *multihomogenization* of F to be

$$F^* = \sum_{n_i} X_{n_i, n_i+1}^{d_{n_i}} F(0, \dots, 0, \mathbf{X}_{n_i}/X_{n_i, n_i+1}, 0, \dots, 0).$$

That is, it is the sum of the homogenizations in each factor of $\mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_r}$. △

4.27: Show that the pole set of a rational function on a variety in any multispace is an algebraic subset.

Proof. Let $V \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$ be a variety and $f/g \in k(V)$. Here f, g are mixed multiforms which are mulihomogeneous of the same multidegree in $X_1, \dots, X_{n_1+\cdots+n_r}$. In particular g is a mixed multiform and so gives a well-defined algebraic subset of the multispace which is precisely the pole set. △

4.28: Let $S : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{n+m+mn}$ be the *Segre embedding*.

(a) Show that S is well-defined and injective.

(b) Show that if W is an algebraic subset of the codomain, its preimage is algebraic in the domain.

(c) Show that the image of S is the algebraic set $V(\{T_{ij}T_{kl} - T_{il}T_{kj} \mid i, k = 0, \dots, n; j, l = 0, \dots, m\})$.

(d) Show that the image of S is a variety.

Proof. Part (a): Well-definedness is obvious. For injectivity, suppose that two pairs in the domain have the same image. Then for any $0 \le i \le n$ we have $[x_i y_0 : x_i y_1 : \cdots : x_i y_m] = x_i[y_0 : \cdots : y_m] = x_i'[y_0' : \cdots : y_m']$. If x_i or x_i' is zero we are done, if not then we may scale by $1/x_i$ and $1/x_i'$ respectively to see that the two elements are the same. Performing this trick for $[x_0 : \cdots : x_m]$ yields the other coordinates equivalent.

Part (b): Suppose W is algebraic and with homogeneous ideal I. On the level of ideals, the Segre embedding is a morphism $k[T_{00}, \dots, T_{nm}] \to k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ with image consisting of all biforms. Then the image of any homogeneous ideal is an ideal of biforms and so gives an algebraic set in the codomain.

Part (c): Note that the image is obviously contained in the given algebraic set, as $T_{ij}T_{kl} - T_{il}T_{kj}$ being zero corresponds to $x_i y_j x_k y_l - x_i y_l x_k y_j$ being zero, and these two products are obviously the same. To see that the image contains the given algebraic set, choose a point $[z_{00} : \cdots : z_{nm}]$ in the algebraic set and

Incomplete △

§ 5 Projective plane curves

5.1 Definitions

5.1: Let F be a projective plane curve. Show that a point P is a multiple point of F if and only if $F(P) = F_X(P) = F_Y(P) = F_Z(P) = 0$.

Proof. Recall that the multiplicity of F is the multiplicity of its dehomogenization F_* with respect to X_i. The multiplicity of F_* is the degree of its smallest form, so the point is multiple if and only if F_* has a degree $k > 1$ form as its smallest form. In this case however the derivative with respect to any variable has smallest form of degree at least one, so evaluation at zero makes each derivative zero. △

5.4 Let P be a simple point of F. Show that the tangent line to F at P has the equation $F_X(P)X + F_Y(P)Y + F_Z(P)Z = 0$ (proof due to ChatGPT).

Proof. Take a projective change of coordinates $P \mapsto [a : b : 1]$. Now dehomogenize F with respect to Z: $f(X, Y) = F(X, Y, 1)$ and expand f at (a, b) so that $f(a + u, b + v) = f(a, b) + f_X(a, b)u + f_Y(a, b)v +$ terms of degree ≥ 2, and the term $f(a, b)$ vanishes. Now the tangent line to f at (a, b) is precisely $f_X(P)u + f_Y(P)v = 0$ and because $u = X - a$, $v = Y - b$ we have $f_X(P)(X-a)+f_Y(P)(Y-b) = 0$. Now the projective closure of this line is the homogenization $f_X(P)(X - aZ) + f_Y(P)(Y - bZ) = f_X(P)X + f_Y(P)Y - (af_X(P) + bf_Y(P))Z = 0$ and the derivatives are the same so $F_X(P)X + F_Y(P)Y - (aF_X(P) + bF_Y(P))Z = 0$. Now Euler's formula yields $F_Z(P)Z = -aF_X(P) - bF_Y(P)Y$, which completes the proof. △

5.5: Let $P = [0 : 1 : 0]$, F a curve of degree n, $F = \sum F_i(X, Z)Y^{n-i}$, F_i a form of degree i. Show that $m_P(F)$ is the smallest m such that $F_m \neq 0$, and the factors of F_m determine the tangents to F at P.

Proof. Recall that $m_P(F)$ is given by $m_P(F_*)$ where we are allowed to dehomogenize at any X, Y, Z. Then dehomogenizing at Y yields $m_P(F) = m_P(\sum F_i)$ and now it is clear that the multiplicity is exactly the smallest m such that $F_m \neq 0$. The tangents to F are those that satisfy $I(P, L \cap F) > m_P(F) = m$. Note that if L is a linear factor of F_m, then $I(P, F \cap L) = I(P, F - F_mY^m \cap L) \geq m_P(F - F_mY^m)m_P(L) > m$, so any factor gives a tangent line. Conversely, a tangent line is one satisfying $I(P, F \cap L) > m$. We may dehomogenize with respect to Y to see $I((0, 0), \sum F_i(X, Z) \cap L(X, 1, Z)) > m$, implying that $L(X, 1, Z)$ is tangent to $F_i(X, Z)$. Therefore it divides the smallest form, namely F_m and the proof is complete. △

5.6: For any F, $P \in F$, show that $m_P(F_X) \geq m_P(F) - 1$.

Proof. Suppose $P = [0 : 0 : 1]$. Then the multiplicities are the degrees of the smallest forms of the dehomogenizations. Now note that $\deg(F)F(X, Y, Z) = F_X(X, Y, Z)X + F_Y(X, Y, Z)Y+F_Z(X, Y, Z)Z$ so $F_X(X, Y, 1) = \deg(F)F(X, Y, 1)-F_Y(X, Y, 1)Y-F_Z(X, Y, 1)$. However the degree of the smallest form of the right-hand side is at least $m_P(F(X, Y, 1))-1$, and this completes the proof. △

5.7: Show that two plane curves with no common components intersect in a finite number of points.

Proof. This is a consequence of the case for affine space: Dehomogenize with respect to each variable and note that each dehomogenization also shares no common components, and so has finite intersection by the affine case in two variables. Then taking a union of the copies of $\mathbb{A}^2$ we cover $\mathbb{P}^2$ with yields a finite set, so a finite intersection. △

5.8: Let F be an irreducible curve.

(a) Show that F_X, F_Y, or F_Z is nonzero.

(b) Show that F has only a finite number of multiple points.

Proof. Part (a): Suppose all of $F_X = F_Y = F_Z = 0$. By Euler's theorem $\deg(F)F = F_XX + F_YY + F_ZZ = 0$, but zero is not irreducible.

Part (b): Suppose F had an infinite number of multiple points. Suppose without loss of generality that $F_X \neq 0$ by (a) and note that $F_X(P) = 0$ for any multiple point. Take the contrapositive of 5.7 to see that we now have that F_X and F have a common component, and $F_X|F$ by the fact that F_X has smaller degree. However this violates irreducibility so we are done. △

5.14: Let $P_1, \ldots, P_n \in \mathbb{P}^2$. Show that there are an infinite number of lines passing through P_1 but not $P_2, \ldots, P_n$. If P_1 is simple on F, we may take these lines to be transversal to F.

Proof. Incomplete △

5.15: Let C be an irreducible projective plane curve and $P_1, \ldots, P_n \in \mathbb{P}^2$ simple points on C, $m_1, \ldots, m_n$ integers. Show that there is a $z \in k(C)$ such that $\mathrm{ord}_{P_i}(z) = m_i$.

Proof. Let $z = (\prod L_i^{m_i})/L_0^{\sum m_i}$ for L_i each containing P_i but avoiding all other P_j and L_0 avoiding all P_i (this is possible by 5.14 and 4.25). This satisfies the order requirement by construction, as we may take as a uniformizing parameter L_i. △

5.16: Let F be an irreducible curve in $\mathbb{P}^2$. Suppose $I(P, F \cap Z) = 1$ and $P \neq [1:0:0]$. Show that $F_X(P) \neq 0$.

Proof. Suppose $F_X(P) = 0$. Then Euler's theorem states that $\deg(F)F = F_X X + F_Y Y + F_Z Z = F_Y Y + F_Z Z$. Now dehomogenize at $Y = 1$ so that $\deg(F)F_*(P) = F_Y(P) + F_Z(P)P_3 = 0$ and note that because $P \in Z$, $F_Y(P) = 0$. Therefore $\deg(F)F = F_Z Z$. However this implies that F dehomogenized at X or Y has a least form of degree at least two, contradicting the intersection number claim. △

5.2 Linear systems of curves