An introduction to the Orlik-Solomon algebra

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Abstract

In this survey we give an overview of the Orlik-Solomon algebra in the context of hyperplane arrangements and, more generally, in matroid theory. In the realizable case we discuss connections between the Orlik-Solomon algebra and the cohomology of the arrangement complement. For the non-realizable case we discuss connections to tropical cohomology of the Bergman fan, as well as generalizations of matroids that axiomatize the combinatorics of general affine arrangements.

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§ 1 Introduction

This survey introduces the *Orlik-Solomon algebra* of an arrangement of hyperplanes, and then of a matroid. This algebra was defined by Orlik and Solomon in 1980, answering postively a conjecture of Arnold relating the de Rham cohomology of the *arrangement complement* to the combinatorics of the arrangement itself. Since that time the Orlik-Solomon algebra has been studied thoroughly in the context of arrangements. We survey a small number of developments in the Orlik-Solomon algebra in the context of arrangements, before turning to the case of *matroids*, combinatorial structures that abstract specific arrangements of hyperplanes.

Our introduction will follow Richard Stanley's *An Introduction to Hyperplane Arrangements*, which can be found in [MRS07].

1.1 Hyperplane arrangements

Let K be a field. Considering K^d as a vector space over K, we define a *hyperplane* in K^d to be a codimension one linear subspace of K^d. We identify hyperplanes with nonzero linear functionals $\alpha \in (K^d)^*$:

$$H = \{u \in k^d \mid \alpha(u) = c_i\}.$$

Using the dot product we may identify $\alpha \in (K^d)^*$ with $\alpha \in K^d$, and we refer to α as the *normal vector of* H.

An *arrangement of hyperplanes* $\mathscr{A} = \{H_1, \ldots, H_n\}$ is a finite set whose elements are hyperplanes. We will write α_i for the normal vector of H_i. An arrangement is said to be *central* if $\bigcap_{H_i \in \mathscr{A}} H_i \neq \emptyset$, or, equivalently, if $c_i = c_j$ for all i, j.

We consider two operations on an arrangement $\mathscr{A}$. Fix some $H' \in \mathscr{A}$, and let $\mathscr{A}' = \mathscr{A} - H'$ and $\mathscr{A}'' = \{H' \cap H_i \mid H_i \in \mathscr{A}\}$. We call $\mathscr{A}'$ the *deletion* with respect to H and $\mathscr{A}''$ the *restriction* with respect to H.

Associated to an arrangement is the *intersection poset* $L(\mathscr{A}) = (X, \leq)$ of nonempty hyperplane intersections. Here we have

$$X = \left\{ \bigcap_{H_i \in A} H_i \mid A \subseteq \mathscr{A}, \bigcap_{H_i \in A} H_i \neq \emptyset \right\},$$

and the partial order is that of *reverse-inclusion*: For $x, y \in X$ we have $x \leq y$ if and only if $x \supseteq y$. We will often abuse notation and write $x \in L(\mathscr{A})$ instead of $x \in X$.

The intersection poset of an arrangement has a number of nice properties, which we list now:

- $L(\mathscr{A})$ is graded of rank $\dim \operatorname{span}\{\alpha_i \mid H_i \in \mathscr{A}\}$ with rank function $\operatorname{rk}_{\mathscr{A}}(x) = \operatorname{codim}(x)$.
- $L(\mathscr{A})$ is a meet-semilattice: For any $x, y \in L(\mathscr{A})$, $x \wedge y$ exists and is unique.
- $\mathscr{A}$ is central if and only if $L(\mathscr{A})$ is a *geometric lattice*.

From the intersection poset we derive our first invariant in the study of hyperplane arrangements, the *characteristic polynomial*. Let $\mu(x) = \mu(\hat{0}, x)$ denote the Möbius function of $L(\mathscr{A})$ evaluated with respect to the bottom element of our poset. Then we define the characteristic polynomial by

$$\chi_{\mathscr{A}}(q) = \sum_{x \in L(\mathscr{A})} \mu(x) q^{\dim(x)}.$$

Example 1.1.1

The *braid arrangement* in $\mathbb{C}^d$ has hyperplanes $H_{ij} = \{v \in \mathbb{C}^d \mid v_i = v_j\}$ and is given by the set

$$\mathcal{B}_d = \{H_{ij} \mid i \neq j\}.$$

For any H_{ij}, the normal vector of H_{ij} may be given by $\alpha_{ij} = ce_i - ce_j$ for any $c \in \mathbb{C}$.

This arrangement is central, with intersection given by the subspace $\bigcap_{H_i \in \mathscr{A}} H_i = \operatorname{span}((1, 1, \ldots, 1))$. Therefore its intersection poset is a lattice.

1.2 Matroids

Matroids abstract the combinatorics of central arrangements to abstract simplicial complexes. Matroids have a famously large number of "cryptomorphisms," of which we state only one.

A *matroid* of rank r on the ground set E is a pair $M = (E, \mathcal{I})$ satisfying the following axioms:

(i) $\mathcal{I}$ is a nonempty (abstact) *simplicial complex*, i.e., $\mathcal{I} \neq \emptyset$ and if $J \in \mathcal{I}$ and $I \subset J$, then $I \in \mathcal{I}$.

(ii) For all $T \subseteq E$, the maximal elements of $\mathcal{I} \cap 2^T$ have the same cardinality.

A matroid comes equipped with a rank function on 2^E given by

$$\mathrm{rk}_M(X) = \max\{|I| \mid I \subseteq X, I \in \mathcal{I}\}.$$

That is, the rank of a subset X is the size of the largest independent set contained within X.

Now we proceed to define the notion of a *flat*, which will be important in connecting matroids to (central) hyperplane arrangements. First, we let $\mathrm{cl}(X) = \{x \in E \mid \mathrm{rk}_M(X \cup \{x\}) = \mathrm{rk}_M(X)\}$ be the *closure* of a set. A *flat* of a matroid, then, is a set X that is closed: $\mathrm{cl}(X) = X$.

The flats of a matroid form the so-called *lattice of flats of* M, $L(M)$, a poset whose elements are flats under the partial order of inclusion. Note that this is indeed a lattice with maximum element $\hat{1} = E$ and minimum element $\hat{0} = \mathrm{cl}(\emptyset)$.

To connect matroids to hyperplane arrangements, we must consider only *simple* matroids. An element $x \in E$ is said to be a *loop* of a matroid if $\mathrm{rk}_M(x) = 0$, and a pair $x, y \in E$ is said to be a *pair of parallel points* of a matroid if $\mathrm{rk}_M(\{x, y\}) = 1$. Simple matroids are precisely those matroids without loops and without parallel points. When M is simple we have that $L(M)$ is *geometric*, a notion we make precise now.

A graded lattice L is *geometric* if it satisfies

(i) L is *atomic*: For any $x \in L$ we may write $x = \bigvee_{j \in J} a_j$, where J is some indexing set and a_j are all *atoms*, elements of rank one.

(ii) L is *submodular*: For any $x, y \in L$ we have $\mathrm{rk}_L(x) + \mathrm{rk}_L(y) \geq \mathrm{rk}_L(x \vee y) + \mathrm{rk}_L(x \wedge y)$.

We have the following theorem connecting arrangements, lattices, and matroids:

Theorem 1

The following are equivalent

(i) L is a geometric lattice.

(ii) $L \cong L(M)$ for some simple matroid M.

In particular, if $\mathscr{A}$ is a central arrangement, then $L(\mathscr{A}) \cong L(M_{\mathscr{A}})$ for some simple matroid $M_{\mathscr{A}}$.

The last statement can be made more explicit. Let A be the set of normal vectors to $\mathscr{A}$, and let $M_{\mathscr{A}}$ be the matroid whose ground set is A and whose independent sets are given by linearly independent subsets of A. Then one has the isomorphism

$$\phi : L(\mathscr{A}) \to L(M_{\mathscr{A}}), \quad \bigcap_{i \in J} H_i \mapsto \mathrm{cl}\left(\{\alpha_i \mid i \in J\}\right).$$

§ 2 The Orlik-Solomon algebra of a hyperplane arrangement

Before describing the work of Arnold and Orlik and Solomon, we recall some basic notions from module theory and differential topology.

The content of this section draws primarily from [Yuz01], with the exposition of Arnold's work taken from [Arn69].

2.1 Tensors, the exterior algebra, and de Rham cohomology

The *tensor product* $V \otimes_K W$ of two K-vector spaces V and W is a vector space whose elements are elements of the free group $F(V \times W)$ modulo K-bilinearity. We construct the *tensor algebra* of an K-vector space V by

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} T^k V = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots.$$

Then the *exterior algebra* of V is the quotient algebra given by

$$\bigwedge(V) = \mathcal{T}(V)/I, \qquad I = \langle v \otimes v \rangle.$$

Now we turn to the *de Rham cohomology* of a smooth manifold M. We define a *smooth differential k-form* on M to be a smooth section of $\Omega^k(M) = \bigwedge^k T^*M$, where T^*M is the cotangent bundle on M. We will refer to these as k-forms. We form the *de Rham complex* by considering the cochain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

where d is the *exterior derivative*. Then one defines the kth *de Rham cohomology* group by considering the quotient

$$H^k_{\mathrm{dR}}(M) = \{\alpha \in \Omega^k(M) \mid d\alpha = 0\}/\{\alpha \in \Omega^k(M) \mid \alpha = d\beta \text{ for some } \beta \in \Omega^{k+1}(M)\}.$$

Here we always take cohomology groups with coefficients in $\mathbb{Z}$.

2.2 Arrangement complements: Motivation

Having developed the basic notions for hyperplane arrangements and matroids, we now turn to studying the *arrangement complement*. For an arrangement $\mathscr{A}$ in K^d, we write $M(\mathscr{A}) = K^d - \left(\bigcup_{H \in \mathscr{A}} H\right)$ for the set-theoretic complement of the ambient space by our collection of hyperplanes.

Orlik and Solomon's approach was influenced by the work of Arnold on the *pure braid group*, the fundamental group of the unordered configuration space. For a topological space X the *ordered configuration space on d points*, $\mathrm{Conf}_d(X)$, is given by $\mathrm{Conf}_d(X) = \{(x_1, \dots, x_d) \in X^d \mid x_i \neq x_j \text{ for any } i \neq j\}$. The *unordered configuration space* is similarly defined by $\mathrm{UConf}_d(X) = \mathrm{Conf}_d(X)/S^d$, the quotient of the ordered configuration space by the natural action of S^d, the symmetric group on d elements.

One then defines the *pure braid group on d strands*, PBr_d, to be the fundamental group $\pi_1(\mathrm{UConf}_d(\mathbb{C}))$. Arnold noted that one could realize the underlying topological space in a different way, as the complement $\mathrm{UConf}_d(\mathbb{C}) = M(\mathcal{B}_d) = \mathbb{C}^d - \mathcal{B}_d$. This makes the topology of $\mathrm{UConf}_d(\mathbb{C})$ accessible via the theory of hyperplane arrangements.

Arnold considered a specific quotient of the exterior algebra generated by hyperplanes in $\mathcal{B}_d$ and was able to prove that this quotient algebra was isomorphic to the de Rham cohomology of the complement $M(\mathcal{B}_d)$.

2.3 The Orlik-Solomon algebra

The Orlik-Solomon algebra gives a construction that abstracts Arnold's quotient algebra, with the goal of preserving the relationship between generators and the so-called "logarithmic forms."

Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in K^d. Now consider the n-dimensional K-vector space generated by $e_1, \dots, e_n$, and let $E(\mathscr{A})$ denote the exterior algebra of this vector space. We canonically identify $e_i \sim H_i$, sometimes writing $e_{H_i} = e_i$. Importantly, $E(\mathscr{A})$ is a commutative graded differential algebra with grading determined by $E^1(\mathscr{A}) = \bigoplus_{i=1}^n K e_i$ and $E^p(\mathscr{A}) = \bigwedge^p E^1(\mathscr{A})$. The differential ∂ is uniquely determined by $\partial e_i = 1$ for all i and the graded Leibniz rule, which states that for any $a, b \in E^k(\mathscr{A})$

$$\partial(ab) = \partial(a)b + (-1)^k \partial(b)a.$$

When considering subsets $S \subseteq [n]$, we will often abuse notation and write e_S in place of $\prod_{i\in S} e_i$ as well as writing $\cap S$ in place of $\bigcap_{i\in S} H_i$. We say a subset $S \subseteq [n]$ is *dependent* if $\cap S \neq \emptyset$ and satisfies that $\{\alpha_i \mid i \in S\}$ is linearly dependent.

We define the *Orlik-Solomon ideal*, $I(\mathscr{A})$, to be the ideal generated by elements e_S such that $\cap S = \emptyset$ and elements $\partial e_{S'}$ such that S' is dependent. Then the *Orlik-Solomon algebra of* $\mathscr{A}$ is given by the quotient algebra

$$\mathrm{OS}(\mathscr{A}) = E(\mathscr{A})/I(\mathscr{A}).$$

We note that the Orlik-Solomon ideal can be generated by a more restricted class of generators, e_S with $S = \emptyset$ and ∂e_U with U minimally dependent. We call these minimally dependent sets *circuits*, and the fact that any ∂e_S with S dependent is a combination of ∂e_U with U a circuit follows by a straightforward application of the Leibniz rule.

2.4 Proving the Orlik-Solomon theorem

Our proof is taken from [Yuz01].

Theorem 2 (Orlik-Solomon)

Suppose $\mathscr{A}$ is an arrangement in $\mathbb{C}^d$. Then

$$\mathrm{OS}(\mathscr{A}) \cong H_{\mathrm{dR}}(M(\mathscr{A})).$$

In order to prove this theorem we will need a lemma regarding the relationship between $\mathrm{OS}(\mathscr{A}')$, $\mathrm{OS}(\mathscr{A})$, and $\mathrm{OS}(\mathscr{A}'')$, where we take the deletion and restriction with respect to some distinguished $H' \in \mathscr{A}$. Let $\lambda : \mathscr{A}' \to \mathscr{A}''$ be the map $\lambda(H) = H \cap H_n$, and note that this induces a map $\lambda : 2^{\mathscr{A}'} \to 2^{\mathscr{A}''}$ given by $\lambda(S) = \{H \cap H_n \mid H \in S\}$.

Lemma 3

Let i be the natural embedding of $\mathrm{OS}(\mathscr{A}')$ into $\mathrm{OS}(\mathscr{A})$ and $j : \mathrm{OS}(\mathscr{A}) \to \mathrm{OS}(\mathscr{A}'')$ the map $j(e_S) = e_{\lambda(S-n)}$. Then the sequence

$$0 \to \mathrm{OS}(\mathscr{A}') \xrightarrow{i} \mathrm{OS}(\mathscr{A}) \xrightarrow{j} \mathrm{OS}(\mathscr{A}'') \to 0$$

is exact.

2.4.1 A long exact sequence for the cohomology of the complement

First some notation. Write $M = M(\mathscr{A})$, $M' = M(\mathscr{A}')$, and $M'' = M(\mathscr{A}'')$. It is clear that $M'' = H_n - \bigcup_{i\neq n} H_n \cap H_i$. The goal of this section will be to establish an exact sequence relating the cohomology of M', M, and M''. To do so we will study fibres over H_n and apply the Thom isomorphism as well as excision to the long exact sequence of a pair.

Consider a tubular neighborhood N of M'' in M' and a fibre bundle $N \to M''$ with fibre homeomorphic to $\mathbb{C}$. We may restrict this bundle to $N_0 = N - M''$ and form a new bundle $N_0 \to M''$ with fibre homeomorphic to $\mathbb{C}^*$. Because these two bundles are restrictions of the trivial bundle on H_n, it follows that they are trivial and so $N = M'' \times \mathbb{C}$ and $N_0 = M'' \times \mathbb{C}^*$ up to homeomorphism.

Therefore we conclude that $H^*(N, N_0) = H^*(M'') \otimes H^*(\mathbb{C}, \mathbb{C}^*)$. Noting that $H^2(\mathbb{C}, \mathbb{C}^*) \cong \mathbb{Z}$, we let t be the generator of $H^2(\mathbb{C}, \mathbb{C}^*)$ corresponding to the natural orientation of $\mathbb{C}$.

Multiplication on the left by t defines a group isomorphism (the *Thom isomorphism*)

$$\tau : H^*(M'') \to H^{*+2}(N, N_0).$$

Since $M = M' - M''$ we consider the excision of $U = M' - N$ from (M', M). The excision isomorphism $\alpha : H^*(M', M) \to H^*(N, N_0)$ we find from τ an isomorphism

$$\alpha^{-1}\tau : H^*(M'') \to H^{*+2}(M', M).$$

Then this isomorphism substituted into the long exact sequence of the pair (M', M) yields

$$\ldots \to H^p(M') \xrightarrow{i^*} H^p(M) \xrightarrow{\tau^{-1}\alpha\delta} H^{p-1}(M'') \xrightarrow{j^*} H^{p+1}(M') \to \ldots \qquad (*)$$

where i and j are the natural embeddings of M into M' and $(M', \emptyset)$ into (M', M), respectively. In the above $\delta : H^p(M) \to H^{p+1}(M', M)$ is the connecting homorphism. We will write $\delta' = \tau^{-1}\alpha\delta$ for the rest of the proof.

2.4.2 Cohomology classes

The goal of this section is to develop an understanding of cohomology classes corresponding to hyperplanes such that they may be related to the logarithmic forms of Arnold.

If H is a hyperplane in V, then $V - H$ is homotopy equivalent to $\mathbb{C}^*$ by projection along H. Then $H^1(V-H)$ is generated by a canonical element β_H corresponding to the natural orientation of $\mathbb{C}^*$. If i_{H_m} is the natural embedding of M into $V - H_m$, we let $\beta_m = i^*_{H_m}(\beta_{H_m}) \in H^1(M)$. These cohomology classes are called *basic cohomology classes* of $H^*(M)$. Likewise we may define basic cohomology classes $\beta'_m \in H(M')$ and $\beta''_m \in H^1(M'')$. We have the following properties regarding basic cohomology classes for all p and any $1 \leq m \leq n-1$.

(i) $i^*(\beta'_m) = \beta_m$.

(ii) $\delta(\beta_m) = 0$.

(iii) $\delta(a\beta_m) = \delta(a)\beta'(m)$ for all $a \in H^*_{\mathrm{dR}}(M)$.

(iv) $\tau^{-1}(\alpha(\beta'_m)t) = \beta''_m$.

(v) $\alpha(\delta(\beta_n)) = t$.

It is convenient (and somewhat natural, thanks to Arnold) to write these basic cohomology classes as differential forms. The canonical generator of $H^1(\mathbb{C}^*)$ can be represented by the logarithmic form $\frac{1}{2\pi\sqrt{-1}}\frac{dz}{z}$. Pulling this form back to M we may write β_m as

$$w_m = \frac{1}{2\pi\sqrt{-1}}\frac{d\alpha_m}{\alpha_m}$$

for $\alpha_m \in (\mathbb{C}^*)^d$ the linear form defining H_m. For $S = \{i_1, \ldots, i_p\}$ we write $w_S = w_{i_1} \wedge \cdots \wedge w_{i_p}$ and $\partial w_S = \sum_{j=1}^p (-1)^{j-1} w_{S_j}$ for $S_j = S - i_j$.

It follows that if $\cap S = \emptyset$ then $w_S = 0$, as there exist $c_j \in \mathbb{C}$ such that $\sum_{j=1}^p c_j \alpha_{i_j} = 1$ and so differentiating we find $\sum_{j=1}^p c_j d\alpha_{i_j} = 0$.

We also have that if S is dependent, then $\partial w_S = 0$. This follows by supposing without loss of generality that $\alpha_{i_p} = \sum_{j=1}^{p-1} c_j \alpha_{i_j}$ for some $c_j \in \mathbb{C}$. Then we may write $w_{i_p} = \sum_{j=1}^{p-1} \frac{c_j \alpha_{i_j}}{\alpha_{i_p}} w_{i_j}$ and applying the formula for ∂w_S shows the claim.

The above results apply also to our basic cohomology classes by the de Rham theorem. At this point we should be relatively confident that the isomorphism we desire exists by the construction of $\mathrm{OS}(\mathscr{A})$ and the fact that our basic cohomology classes vanish under the same suppositions as those that define the Orlik-Solomon ideal.

2.4.3 An isomorphism between the short exact sequences

Consider the ring homomorphism $\varphi : E(\mathscr{A}) \to H^*(M)$ given by $e_m \to \beta_m$. To show that ψ is an isomorphism we induct on $\mathscr{A}$. If $\mathscr{A} = \emptyset$ then $\mathrm{OS}(\mathscr{A}) \cong H^*(M) \cong \mathbb{Z}$. Now suppose that $\mathscr{A} \neq \emptyset$, and consider the induced homomorphism $\psi : \mathrm{OS}(\mathscr{A}) \to H^*(M)$, which allows us to construct the following diagram for all p.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{OS}(\mathscr{A}') & \longrightarrow & \mathrm{OS}(\mathscr{A}) & \longrightarrow & \mathrm{OS}(\mathscr{A}'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi''} & & \\
0 & \longrightarrow & H^p(M') & \xrightarrow{i^*} & H^p(M) & \xrightarrow{\delta'} & H^{p+1}(M'') & \longrightarrow & 0
\end{array}$$

The commutativity of this diagram follows by properties (i)-(v) in 2.4.2. Both ψ'' and ψ' are isomorphisms by supposition, and so the five lemma gives that ψ is an isomorphism

§ 3 The Orlik-Solomon algebra of a matroid

One should note that the Orlik-Solomon algebra's definition makes reference only to notions of emptiness or linear dependence, which implies that for central arrangements it may be described by the underlying matroid, $M_{\mathscr{A}}$. Further, one might ask for a generalization of matroids that axiomatizes the combinatorics of not-necessarily-central arrangements so that the Orlik-Solomon algebra may be further generalized. In this section we discuss the Orlik-Solomon algebra's combinatorial consequences, including its implications for the lattice of flats and its definition for arbitrary matroids.

3.1 Homology of the lattice of flats

Our first purely combinatorial interpretation of the Orlik-Solomon algebra has to do with the intersection poset. Again, however, the Orlik-Solomon algebra's definition requires no geometric context, and so the results that follow hold generally for any meet-semilattice, including the lattice of flats of a matroid. The content of this section comes from [Yuz01].

Fix a meet-semilattice L. We write $A(L)$ for the set of atoms of L and let $n = |A(L)|$. We write $\hat{0}$ and $\hat{1}$ for the minimum and maximum (if it exists), respectively. We may choose an arbitrary linear order on $A(L)$ such that there is a canonical identification $A(L) \to [n]$. We say $\sigma \subseteq [n]$ is *bounded* if $\bigvee \sigma$ exists, and *unbounded* otherwise.

We introduce two homology theories on L. The *flag complex* of L is the abstract simplicial complex whose simplices are chains in L, and we write $H_*(L)$ for the homology of the flag complex. The *relative atomic complex*, $D(L)$, is a chain complex given by the free R-module with basis consisting of all sets $\sigma \subset [n]$ bounded in L. There is a grading on $D(L)$ given by $|\sigma|$, and the differential d on $D(L)$ is defined by

$$d(\sigma) = \sum_{\{j \in \sigma \mid \bigvee \sigma_j = \bigvee \sigma\}} (-1)^{j-1} \sigma_j,$$

where $\sigma = \{i_1, \ldots, i_p\}$ and $\sigma_j = \sigma - i_j$.

For $L_X = \{Y \in L \mid Y \leq X\}$ we let $D'(X)$ be the chain complex of the *atomic complex* of L_X, the complex whose simplices are subsets bounded in $L_X - \hat{1}$. Let $D''(X)$ be the chain complex of the simplex with vertex set $A(L_X)$. We state now some important facts regarding the two homology theories.

(i) $D(L) = \bigoplus_{X \in L} D(X)$, where $D(X)$ is generated by $\sigma \subset [n]$ such that $\bigvee \sigma = X$.

(ii) For any p and $X \in L$, the group $H_p(D(X))$ is naturally isomorphic to $H_{p-2}(L_X)$.

(iii) If L_X is a geometric lattice for all X, then $H_p(D(X)) = 0$ for all $p \neq \mathrm{rk}_L(X)$.

Property (i) is fairly obvious. For property (ii), note that we have the short exact sequence

$$0 \to D'(X) \to D''(X) \to D(X)_{(+1)} \to 0$$

and the property follows from the corresponding long exact sequence of homology using the acyclicity of $D''(X)$. Property (iii) follows from (ii) and Folkman's theorem [Fol66].

Property (iii) allows us to consider generators of $H_*(D(X))$ in the case that $L(X)$ is a geometric lattice, which is the case for both matroids and arrangements of hyperplanes. For $\sigma \subset [n]$ we say that σ is *independent* if if is bounded and satisfies $\bigvee \sigma_j < \bigvee \sigma$ for all $j \in \sigma$. Note that this coincides with the definition of independent (that is, not dependent) for the Orlik-Solomon algebra in 2.3. Regardless, each independent set is a cycle in D and so determines a homology class $\zeta_\sigma \in H_{|\sigma|}(D(X))$ for $X = \bigvee \sigma$.

If L is a geometric lattice then $\mathrm{rk}_L(\bigvee \sigma) \leq |\sigma|$, and so each set of atoms σ such that $\bigvee \sigma = X$ and $|\sigma| = \mathrm{rk}_L(X)$ is independent. That is, $D_{\mathrm{rk}_L(X)}(X)$ consists only of cycles, and $H_*(D)$ is generated by ζ_σ with σ independent.

Theorem 5

The correspondence $e_i \mapsto \zeta_i$ generates an isomorphism of algebras $\mathrm{OS}(\mathscr{A}) \to H_*(D)$.

Proof. We note that the map $e_i \mapsto \zeta_i$ generates a surjective algebra homomorphism $\varphi : E(\mathscr{A}) \to H_*(D)$, and that if S is unbounded or dependent then $\varphi(e_S) \in \ker \varphi$. Now consider an R-linear combination of some e_S with independent S. This sum lies in $\ker \varphi$ if and only if it is a sum of elements whose images under φ are boundaries in some $D(X)$. These elements have the form ∂e_S for S bounded and $\mathrm{rk}_L(\cap S) = |S| - 1$. The rank condition implies that $e_S = \pm e_T e_U$ where T is independent and U is a circuit. Then $\partial e_S = \pm e_T(\partial e_U) + (\partial e_T)e_U$ by the Leibniz rule. The first summand contains ∂e_U, a dependent element, so it is sent to zero. The second summand also must lie in $\ker \varphi$ as we have $\partial e_S \in \ker \varphi$. Therefore $\ker \varphi$ is generated by monomials e_S with unbounded S and elements $e_T(\partial e_U)$ with U a circuit. This implies that $\ker \varphi = I(\mathscr{A})$, completing the proof. △

3.2 The Bergman fan and tropical cohomology

Having developed a connection between the Orlik-Solomon algebra and the homology of a (meet-semi)-lattice, it is time to meet the Orlik-Solomon algebra for a matroid. The content of this section comes from [Zha13].

Let M be a loopless matroid of rank r on the set $\{0, n\}$. Let W be the rank $n+1$ free abelian group with generators $w_0, \ldots, w_n$ canonically identified with elements of the ground set of our matroid. We define the *Orlik-Solomon ideal* of M, $I(M)$, to be the ideal in $\bigwedge W$ generated by elements of the form ∂w_S for S a dependent set in M. Then the *Orlik-Solomon algebra* of M is the quotient algebra

$$\mathrm{OS}(M) = \bigwedge W / I(M).$$

Letting W_0 be the subgroup of W generated by differences of the form $w_i - w_j$, we let $I_0(M)$ be the restriction of $I(M)$ to the subalgebra $\bigwedge W_0 \subset \bigwedge W$. We define the *projective Orlik-Solomon algebra* of M to be

$$\mathrm{OS}_0(M) = \bigwedge W_0 / I_0(M).$$

Supposing that $\mathscr{A}$ is a central arrangement in $\mathbb{C}^d$ and $M_{\mathscr{A}}$ the associated matroid, $\mathrm{OS}_0(M_{\mathscr{A}}) \cong \mathrm{OS}(\mathscr{A})$.

Associated to a matroid is the *Bergman fan*, a balanced polyhedral fan that connects nicely matroid theory and tropical geometry. To define this fan we first define a more general simplicial fan associated to M.

Let V be the rank $n+1$ abelian group on generated by $e_0, \ldots, e_n$ and $\Sigma(M) \subset V_{\mathbb{R}}$ be the simplicial fan on $L(M)$ (that is, the fan whose rays are $e_F = \sum_{i \in F} e_i$ for any flat $F \in L(M)$). Then the *Bergman fan* of M, $\Sigma_0(M)$, is the quotient of $\Sigma(M)$ along $e_M = \sum_{i=0}^n e_i$.

For a integral polyhedral fan in $\mathbb{R}^N$, the *tropical homology* of a fan Σ, $\mathcal{F}_k(\Sigma)$, is the subgroup of $\bigwedge^k \mathbb{Z}^N$ generated by elements of the form $v_1 \wedge \cdots \wedge v_k$, where all $v_i \in \langle \sigma \rangle_{\mathbb{Z}}$, the integral lattice in the vector subspace spanned by a cone $\sigma \in \Sigma$. Naturally, the *tropical cohomology*, $\mathcal{F}^k(\Sigma)$, is the dual group $\mathcal{F}^k(\Sigma) = \mathrm{Hom}(\mathcal{F}_k(\Sigma), \mathbb{Z})$.

A result of Zharkov yields that the Bergman fan has *tropical cohomology* isomorphic to the projective Orlik-Solomon algebra of M.

Theorem 6

There is a canonical isomorphism

$$\varphi : \mathcal{F}^*(\Sigma_0(M)) \to \mathrm{OS}_0(M).$$

§ 4 Further reading

Our discussion of the Orlik-Solomon algebra has noted that there is some natural desire for a generalization of matroids that captures the combinatorics of affine arrangements, or more generally of meet-semilattices L satisfying L_X geometric for all $X \in L$.

In [Kaw04], Kawahara defines the family of *quasi-matroids* with the goal of establishing a matroid analogue that satisfies exactly our request. These are equivalent to Ardila's *semimatroids* [Ard].

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